

Overly determined agents prevent consensus in a generalized Deffuant model on \mathbb{Z} with dispersed opinions

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Abstract

During the last decades, quite a number of interacting particle systems have been introduced and studied in the border area of mathematics and statistical physics. Some of these can be seen as simplistic models for opinion formation processes in groups of interacting people. In the one introduced by Deffuant et al. agents, that are neighbors on a given network graph, randomly meet in pairs and approach a compromise if their current opinions do not differ by more than a given threshold value θ . We consider the two-sidedly infinite path \mathbb{Z} as underlying graph and extend former investigations to a setting in which opinions are given by probability distributions. Similar to what has been shown for finite-dimensional opinions, we observe a dichotomy in the long-term behavior of the model, but only if the initial narrow-mindedness of the agents is restricted.

1 Introduction

The research field that became known as *opinion dynamics* originated from simple models for interacting elementary particles established in statistical physics, introduced to figure out how microscopic interaction rules lead to macroscopic properties of the whole system. Due to the strong link between statistical mechanics and spatial stochastic processes, interest among mathematicians was raised and in the course of a few decades an abundance of new models with similar but qualitatively different interaction schemes was introduced and analyzed, primarily by computer-based stochastic simulations. The survey article [1] gives a broad overview of the different models and their analyses and applications.

Despite their radical limitations in terms of complexity, these models attracted more and more the attention of the social sciences and were used to

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describe group behavior on an elementary level and to explain real life phenomena. In 2000, Deffuant et al. [3] suggested a simple model that features a bounded confidence restriction: Neighbors talk to each other in pairs and their opinions are updated towards a compromise only if the opinions they hold just before they meet are not further apart than a given threshold. This is meant to capture the realistic phenomenon that people tend to modify their attitude on a specific topic when talking to others, but not if they consider the views of their discussion partner as so alien as to seem like complete nonsense.

In mathematical terms, the *Deffuant model* is structured as follows: First, we are given a simple connected graph $G = (V, E)$, that shapes the underlying network. The vertices are understood to represent agents holding individual opinions on a certain topic. The edges of the graph are supposed to represent the connections between these individuals and entail a possible mutual influence among neighbors. The vertex set V can be either finite or countably infinite. In the latter case the maximal degree in G is commonly assumed to be bounded.

Then there are two model parameters: the already mentioned confidence bound $\theta > 0$ and the convergence parameter $\mu \in (0, \frac{1}{2}]$, shaping the step size towards a compromise when two opinions are updated. Opinions usually take values in \mathbb{R} . A higher-dimensional analog was considered in [7] and here we will extend the model further. For these generalizations, we need to specify a metric d that is used to measure the distance of two opinions and takes over the task of the absolute value in the original model.

The first source of randomness is the configuration of initial opinions. Even though there have been attempts to look at settings with dependent initial opinions (see for example Section 2.2 in [6]) the usual setting is to take i.i.d. initial opinions which then evolve dependencies by interacting. The opinion value at vertex $v \in V$ and time $t \geq 0$ will be denoted by $\eta_t(v)$.

The second source of randomness in the model is the succession of pairwise encounters. On finite graphs, the next pair of neighbors to meet is picked uniformly at random. On infinite graphs, the corresponding equivalent is to assign i.i.d. Poisson processes on all edges of the graph: Whenever a *Poisson event* occurs on the edge $e = \langle u, v \rangle$, i.e. a jump in the Poisson process associated with e , the agents u and v interact in the following way: Assume the event happens at time t and the opinions of u and v just before are given by $\eta_{t-}(u) = \lim_{s \uparrow t} \eta_s(u) =: a$ and $\eta_{t-}(v) = \lim_{s \uparrow t} \eta_s(v) =: b$ respectively. Then, depending on the distance of a and b , there might be an update according to the following rule:

$$\eta_t(u) = \begin{cases} a + \mu(b - a) & \text{if } d(a, b) \leq \theta, \\ a & \text{otherwise} \end{cases} \quad (1)$$

and similarly

$$\eta_t(v) = \begin{cases} b + \mu(a - b) & \text{if } d(a, b) \leq \theta, \\ b & \text{otherwise.} \end{cases}$$

Given our assumptions, E is countable, so there will almost surely be neither

two simultaneous Poisson events nor a limit point in time for the Poisson events on edges incident to one fixed vertex. This guarantees the well-definedness of the process by (1) for finite G . The extension to infinite graphs with bounded degree is not immediately obvious but a standard argument, see Thm. 3.9 in [11].

When it comes to the long term behavior of the system, it is quite natural to ask whether the agents will form a consensus to which all the opinions converge or not. Let us properly define and distinguish the following two opposing asymptotics of the Deffuant model as time tends to infinity:

Definition 1

(i) *Disagreement*

There will be finally blocked edges, i.e. edges $e = \langle u, v \rangle \in E$ s.t.

$$d(\eta_t(u), \eta_t(v)) > \theta,$$

for all times t large enough. Hence the vertices fall into different opinion groups, that are incompatible with neighboring ones.

(ii) *Consensus*

The value at every vertex converges, as $t \rightarrow \infty$, to a common limit l , where

$$l = \begin{cases} \frac{1}{|V|} \sum_{v \in V} \eta_0(v), & \text{if } G \text{ is finite} \\ \mathbb{E} \eta, & \text{if } G \text{ is infinite} \end{cases}$$

and $\mathcal{L}(\eta)$ denotes the distribution of the initial opinion values.

Even though these two regimes intuitively seem to be complementary, for infinite graphs it is far from obvious that the asymptotics of the model is necessarily given by one or the other (cf. Def. 1.1 in [6] and also the remark at the end of Section 5).

In this paper, we are going to consider the two-sidedly infinite path \mathbb{Z} as underlying graph, i.e. $V = \mathbb{Z}$ and $E = \{\langle v, v+1 \rangle, v \in \mathbb{Z}\}$. The first result for this setting was published in 2011 and is due to Lanchier [10], who showed a sharp phase transition from almost sure disagreement to almost sure consensus at $\theta = \frac{1}{2}$, given initial opinions, that are i.i.d. $\text{unif}([0, 1])$. Shortly thereafter, Häggström [5] reproved Lanchier's findings using a quite different approach; then Häggström and Hirscher [6] extended them to general univariate distributions for $\mathcal{L}(\eta)$. In [7], the case of vector-valued opinions and different distance measures was examined.

One aspect that could be considered unrealistic in these models is the fact that even though opinions are random, for a fixed realization they were given by numbers or vectors, hence entirely determined – not doing justice to the extremely common phenomenon of uncertainty in people's opinions. In what follows, we are going to introduce and analyze a variant of the Deffuant model on \mathbb{Z} , where the opinions are given by random absolutely continuous measures

on $[0, 1]$. The support of these measure-valued opinions can be seen to represent uncertainty: the more concentrated the measure, the more determined the agent.

As a general preparation for an extension of the model in this direction, in Section 2 we will introduce the total variation distance (which will be used as distance measure) and recall a Strong law of large numbers (SLLN) for continuous densities, due to Rubin, replacing the common SLLN which was a crucial ingredient in the case of finite-dimensional opinions.

The model with measure-valued opinions is outlined in Section 3. We consider random symmetric triangular distributions as initial opinions and find that for this setting overly determined agents (i.e. agents whose initial opinion is concentrated on sufficiently short intervals) prevent consensus for all $\theta \in [0, 1)$, cf. Theorem 3.1.

The results for finite-dimensional opinion spaces listed above will be sketched in more detail in Section 4. Central ideas from [5] will be presented as they prove to be useful in our setting as well.

In Section 5 the main result for the setting with unrestricted symmetric triangular distributions, Theorem 3.1, is proved: We show that the behavior of the model is trivial in this case as extremely determined agents will have and keep a total variation distance close to 1 to their neighbors' opinions.

If we put a restriction on the initial determination of the agents by disallowing triangular distributions that have a support of length less than a fixed value γ , the familiar phenomenon of a phase transition in θ from a.s. disagreement to a.s. consensus reappears. This case, as well as the precise dependency of the threshold value θ_c on the parameter γ are elaborated in Section 6. In the final section, we briefly discuss possible other initial configurations and earlier attempts to incorporate inhomogeneous open-mindedness of the agents.

2 Distance and convergence of absolutely continuous random measures

As indicated, we want to generalize the Deffuant model on \mathbb{Z} further by looking at opinions that are no longer numbers or vectors but probability distributions instead. These random distributions can be seen to shape indeterminacy in the agents: Even with initial opinion profile and sequence of encounters fixed, the opinion of an individual at a given time is not a fixed value but a probability measure. Initially, the agents are independently assigned random measures from a common distribution. When they meet and their current opinion measures do not differ by more than θ , with respect to a fixed metric on probability measures, the new opinions will be given by convex combinations of the old ones, just as described in (1).

In order to quantify the difference between two distributions there are quite a few metrics to choose from. The so-called total variation distance is among the most common ones.

Definition 2

Let μ and ν be two probability distributions on a set S . The *total variation distance* between the two measures is then defined as

$$\|\mu - \nu\|_{\text{TV}} := \sup_{A \subseteq S} |\mu(A) - \nu(A)|.$$

As the total variation distance of two probability distributions is a number in $[0, 1]$, the non-trivial values for θ lie in $(0, 1)$ for this model. Further, note that the total variation distance of two probability distributions μ and ν , that are absolutely continuous with respect to the Lebesgue measure on \mathbb{R} and have densities f and g respectively, is given by

$$\|\mu - \nu\|_{\text{TV}} = \frac{1}{2} \int_{\mathbb{R}} |f(x) - g(x)| \, dx.$$

In addition, if μ and ν are distributions on $[0, 1]$, we can immediately conclude $\|\mu - \nu\|_{\text{TV}} \leq \frac{1}{2} \|f - g\|_{\infty}$, where $\|f\|_{\infty} = \sup_{x \in [0, 1]} |f(x)|$ denotes the supremum norm on $[0, 1]^{\mathbb{R}}$.

To be able to transfer the findings from the Deffuant model on \mathbb{Z} featuring real- or vector-valued opinions, we further need an equivalent for the Strong law of large numbers (SLLN) geared towards the densities of random measures. The following result of Rubin [13] serves our purposes.

Let U, U_1, U_2, \dots denote a sequence of independent, identically distributed random variables with values in an arbitrary space Y . Given a compact topological space X , consider a map $f : X \times Y \rightarrow \mathbb{R}$, that is measurable in the second argument for each $x \in X$.

Theorem 2.1 (SLLN for continuous densities)

If there exists an integrable function g on Y such that $|f_y(x)| < g(y)$ for all $x \in X$ and $y \in Y$, as well as a sequence of measurable sets $(S_i)_{i \in \mathbb{N}}$ with

$$\mathbb{P}\left(U \in \bigcap_{i \in \mathbb{N}} S_i^c\right) = 0,$$

and the property that $\{f_y(\cdot), y \in S_i\}$ is equicontinuous on X for all $i \in \mathbb{N}$, then with probability 1,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n f_{U_i}(x) = \mathbb{E} f_U(x)$$

uniformly in $x \in X$ and the limit function is continuous.

The measure with density $\mathbb{E} f_U$ is commonly called *intensity* or *intensity measure*, see e.g. Section 1.2 in [8] for a more general introduction.

3 Initial opinions given by random triangular distributions

For concreteness, let us pick the initial opinions from a specific class of absolutely continuous distributions. A rather natural choice, departing from real-valued opinions (which can be seen as Dirac delta measures), are symmetric triangular distributions on random subintervals of $[0, 1]$, the endpoints of which are chosen uniformly from $[0, 1]$.

More precisely, let us consider the initial opinions $\{\eta_0(v), v \in \mathbb{Z}\}$ to be picked in the following way: Consider $\{U(v), v \in \mathbb{Z}\}$ to be an i.i.d. sequence of $\text{unif}([0, 1]^2)$ random vectors. The node v will be assigned an initial opinion given by the random absolutely continuous probability measure with density

$$f_0^{(v)}(x) = \begin{cases} 0, & x \notin (m, M) \\ \left(\frac{2}{M-m}\right)^2 \cdot (x-m), & x \in (m, \frac{m+M}{2}] \\ -\left(\frac{2}{M-m}\right)^2 \cdot (x-M), & x \in (\frac{m+M}{2}, M) \end{cases} \quad (2)$$

$$= \frac{2}{|y-z|} \cdot \left(1 - \frac{2}{|y-z|} \cdot \left|x - \frac{y+z}{2}\right|\right)^+, \quad x \in [0, 1], \quad (3)$$

where $U(v) = (y, z)$ and $m := \min\{y, z\}$, $M := \max\{y, z\}$, see Figure 1.

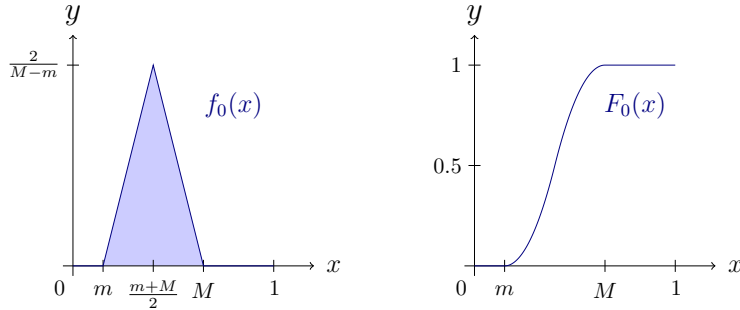


Figure 1: The density and distribution function of a symmetric triangular distribution on $[m, M]$.

Seen from a different angle, to get the initial opinion of a fixed agent, we first choose a central opinion value C uniformly from $[0, 1]$ and then a spread for the support of the distribution uniformly among $[0, \min\{C, 1 - C\}]$. That this procedure is equivalent to the one described above is an immediate consequence of the change of variable formula, see the proof of Lemma 5.1 (especially Figure 3) for more details.

Note that this model features two qualitatively different forms of extreme initial opinions: On the one hand – as in the original model – the agents can have opinions lying at the edges of the spectrum (i.e. concentrated close to 0 or 1 in this case), on the other an individual opinion can be very determined in the

sense that $U(v)$ is close to the diagonal (i.e. $|y - z|$ very small), which provokes a highly concentrated density and necessarily a large distance to a vast majority of possible initial opinions.

This effect is quite realistic: Irrespectively of their opinion being exceptional or mainstream, people that are extremely narrow-minded or determined are usually neither willing to consider the opinion nor to accept the arguments of others, let alone to compromise. In this sense, even though the mathematics are closely related to the case of finite-dimensional opinions, the extension of the model to measure-valued opinions introduces an additional real life phenomenon.

For symmetric triangular distributions on $[0, 1]$ without any restriction on the minimal length of their support, we are going to show that the model exhibits a trivial behavior:

Theorem 3.1

Consider the Deffuant model on \mathbb{Z} , where the initial opinions are given by independently assigned random triangular distributions as described in (3). Then for all $\theta \in [0, 1)$, the system almost surely approaches disagreement in the long run if the total variation distance is used to measure the distance between two opinions.

For the proof of this result, we refer the reader to Section 5. Since this setting allows to reuse results from the finite-dimensional case, we first want to give a brief overview of these in the following section.

4 Background

As mentioned in the introduction, the first analytic result about consensus formation in the Deffuant model on \mathbb{Z} was established by Lanchier [10] and deals with opinion profiles that are initially given by an i.i.d. sequence of $\text{unif}([0, 1])$ random variables. The distance between two opinions was taken to be the absolute value of their difference. Häggström [5] used different techniques to reprove and slightly sharpen this result. His arguments were later adapted to accommodate other univariate initial distributions as well, leading to an analog covering all marginal distributions that have a first moment $\mathbb{E}\eta_0 \in \mathbb{R} \cup \{-\infty, +\infty\}$, see Thm. 2.2 in [6]:

Theorem 4.1

Consider the Deffuant model on the graph (\mathbb{Z}, E) , where $E = \{\langle v, v+1 \rangle, v \in \mathbb{Z}\}$, with fixed parameter $\mu \in (0, \frac{1}{2}]$. Let the initial configuration be given by an i.i.d. sequence of real-valued random variables, having the common distribution $\mathcal{L}(\eta_0)$, and the distance of two opinions by the absolute value of their difference.

- (i) *Given a bounded distribution $\mathcal{L}(\eta_0)$ with expected value $\mathbb{E}\eta_0$, let $[a, b]$ denote the smallest closed interval containing its support. If $\mathbb{E}\eta_0$ does not lie in the support, let $I \subset [a, b]$ denote the maximal, open interval with $\mathbb{E}\eta_0 \in I$ and $\mathbb{P}(\eta_0 \in I) = 0$. In this case, set h to be the length of I , otherwise set $h = 0$.*

Then the critical value for θ , marking the phase transition from a.s. disagreement to a.s. consensus, becomes $\theta_c = \max\{\mathbb{E} \eta_0 - a, b - \mathbb{E} \eta_0, h\}$. The common limit value in the supercritical regime is $\mathbb{E} \eta_0$.

- (ii) Suppose the distribution $\mathcal{L}(\eta_0)$ is unbounded but its expected value exists, i.e. $\mathbb{E} \eta_0 \in \mathbb{R} \cup \{-\infty, +\infty\}$. Then the Deffuant model with arbitrary fixed parameter $\theta \in (0, \infty)$ will a.s. behave subcritically, meaning that disagreement will be approached in the long run.

With an appropriate adaptation to the more involved geometry of vector-valued opinions, the main ideas in [5] further served to establish similar results for the Deffuant model on \mathbb{Z} with opinion space \mathbb{R}^d , $d > 2$, and more general distance measures, see Thm. 3.15 and Thm. 4.11 in [7].

Since the same line of reasoning was used in both [5], [6] and [7] to derive the results for finite-dimensional opinion spaces we just mentioned, let us now take a closer look on the used key concepts and crucial auxiliary results. They form the base for most of the conclusions we will be able to draw in the case of infinite-dimensional opinions.

In [5], Häggström presents two central ideas, whose effect turns out to be highly limited to paths, but combined they prove to be quite powerful in the analysis of the Deffuant model on the infinite path \mathbb{Z} . The first one is the notion of *flat points*:

Definition 3

Consider the initial i.i.d. configuration $\{\eta_0(u)\}_{u \in \mathbb{Z}}$ with common marginal distribution $\mathcal{L}(\eta_0)$ on an opinion space X (e.g. \mathbb{R} or \mathbb{R}^d), which we consider to be equipped with the metric ρ . Under the premise that the mean $\mathbb{E} \eta_0$ of the initial distribution exists and given $\varepsilon > 0$, a vertex $v \in \mathbb{Z}$ is called *ε -flat to the right* (with respect to the initial configuration), if for all $n \geq 0$:

$$\frac{1}{n+1} \sum_{u=v}^{v+n} \eta_0(u) \in B_\varepsilon(\mathbb{E} \eta_0), \quad (4)$$

where $B_r(x) := \{y \in X, \rho(x, y) \leq r\}$ denotes the (closed) ρ -ball around $x \in X$ with radius $r > 0$. A vertex v is called *ε -flat to the left* if the above condition is met with the sum running from $v-n$ to v instead. Finally, v is called *two-sidedly ε -flat* if for all $m, n \geq 0$

$$\frac{1}{m+n+1} \sum_{u=v-m}^{v+n} \eta_0(u) \in B_\varepsilon(\mathbb{E} \eta_0). \quad (5)$$

The crucial role vertices, that are one- or two-sidedly ε -flat with respect to the initial configuration, can play in the further evolution of the configuration becomes more obvious in the light of the second key idea, the non-random pairwise averaging procedure Häggström [5] proposed to call *Sharing a drink* (SAD) on \mathbb{Z} .

Glasses are put along the infinite path at all integers; the one at site 0 is full, all others are empty. Similarly to the Deffuant model, neighbors interact and share, but now we skip randomness and confidence bound: The procedure starts with the initial profile $\{\xi_0(v)\}_{v \in \mathbb{Z}}$, given by $\xi_0(0) = 1$ and $\xi_0(v) = 0$ for all $v \neq 0$. In each step, we choose an edge, along which an update of the form (1) is executed; more precisely, if we are given the profile $\{\xi_n(v)\}_{v \in \mathbb{Z}}$ after step n and choose $\langle u, u+1 \rangle$ for the next round, we get

$$\begin{aligned}\xi_{n+1}(u) &= (1 - \mu) \xi_n(u) + \mu \xi_n(u+1), \\ \xi_{n+1}(u+1) &= \mu \xi_n(u) + (1 - \mu) \xi_n(u+1), \\ \xi_{n+1}(v) &= \xi_n(v) \quad \text{for all } v \notin \{u, u+1\}.\end{aligned}\tag{6}$$

The resulting profiles, after we have performed this procedure a finite number of rounds, will be called *SAD-profiles*. Besides the facts that they feature only finitely many non-zero elements, the elements are all positive and sum to 1, there are less obvious properties that these profiles share which we will collect in the following lemma (for proofs, see Lemmas 2.2, 2.1 and Thm. 2.3 in [5]):

Lemma 4.2

Consider the SAD-procedure on the infinite path \mathbb{Z} , started in vertex v , i.e. with $\xi_0(u) = \delta_v(u)$, $u \in V$. Then we get the following:

- (i) *All achievable SAD-profiles are unimodal.*
- (ii) *If the vertex v only shares the water to one side, it will remain a mode of the SAD-profile.*
- (iii) *The supremum over all achievable SAD-profiles started with δ_v at another vertex w equals $\frac{1}{d+1}$, where d is the graph distance between v and w .*

The connection to the Deffuant model is established in Lemma 3.1 in [5]: The opinion value $\eta_t(0)$ at any given time $t > 0$ can be written as a weighted average of the initial opinions, where the weights are given by the (random) SAD-profile which is dual to the dynamics in the Deffuant model in the sense that the order of updates has to be reversed.

Combining this link with the concept of ε -flatness makes it possible to derive the following crucial auxiliary results (which are obvious generalizations of intermediate results, established in the proofs of Prop. 5.1, as well as of Lemma 6.3 in [5]):

Lemma 4.3

Consider the Deffuant model on \mathbb{Z} with initial configuration be given by an i.i.d. sequence of random variables having the common distribution $\mathcal{L}(\eta_0)$.

- (i) *If vertex v is ε -flat to the right with respect to the initial configuration and does not interact with vertex $v-1$, its opinion stays inside $B_\varepsilon(\mathbb{E} \eta_0)$. The same holds for ε -flatness to the left and $v+1$ in place of $v-1$.*
- (ii) *If vertex v is two-sidedly ε -flat with respect to the initial configuration, its opinion value will stay inside $B_{6\varepsilon}(\mathbb{E} \eta_0)$, irrespectively of the dynamics.*

Without much further work, these findings can be used to analyze the behavior of the model featuring unrestricted symmetric triangular distributions, as we will see in the following section.

5 Overly determined agents prevent consensus

As the expectation of the initial distribution played a central role in the model featuring real- or vector-valued opinions, we first have to get our hands on its counterpart in the context of random measures, the intensity, before we can set about proving Theorem 3.1.

Lemma 5.1

Consider the absolutely continuous random measure η to be given by the density

$$f_U(x) = \frac{2}{|y-z|} \cdot \left(1 - \frac{2}{|y-z|} \cdot \left|x - \frac{y+z}{2}\right|\right)^+, \quad x \in [0, 1], \quad (7)$$

where $U = (y, z)$ is taken uniformly from the unit square $[0, 1]^2$, as introduced in (3). Then its intensity measure (commonly denoted $\mathbb{E}\eta$) is given by the density

$$\varphi(x) = \begin{cases} -8 \left[(1-x) \ln(1-x) + x (1 - \ln(2)) \right], & x \in [0, \frac{1}{2}] \\ -8 \left[x \ln(x) + (1-x) (1 - \ln(2)) \right], & x \in [\frac{1}{2}, 1] \end{cases}.$$

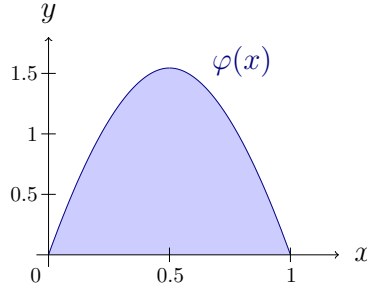


Figure 2: Density of the intensity measure corresponding to random symmetric triangular distributions on $[0, 1]$.

PROOF: Fix $x \in (0, 1)$. First of all, by symmetry, we can take U to be uniform on the set $A := \{(y, z) \in \mathbb{R}^2, 0 \leq z \leq y \leq 1\}$. To further simplify the calculations, let us consider the simple linear transform $T((y, z)) = \frac{1}{2}(y+z, y-z)$, depicted in Figure 3 below. From the change of variable formula we know that $T(U)$ is uniform on the set $B := \{(u, v), v \in [0, \frac{1}{2}], u \in [v, 1-v]\}$.

Given the random density f_U as in (7), we can write

$$f_{T(U)}(x) = \frac{1}{v} \cdot \left(1 - \frac{1}{v} \cdot |x - u|\right)^+, \quad x \in [0, 1]$$

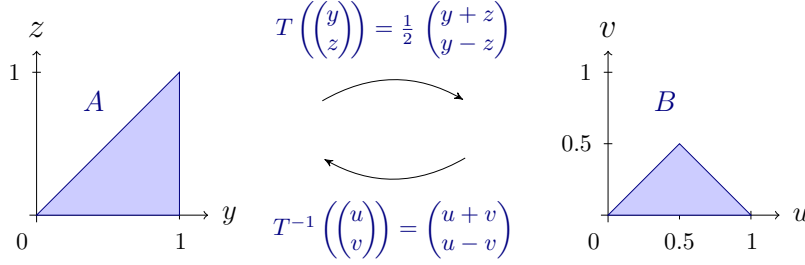


Figure 3: Using transform T to consider the arithmetic mean and half the distance of two independent $\text{unif}([0, 1])$ random variables instead.

and conclude that $f_{(u,v)}(x)$ is non-zero for (u, v) in $B_x = B_1 \cup B_2 \cup B_3$, where

$$B_1 := \{(u, v), v \in [0, \frac{x}{2}], u \in [x - v, x + v]\}$$

$$B_2 := \{(u, v), v \in [\frac{x}{2}, \frac{1-x}{2}], u \in [v, x + v]\}$$

$$B_3 := \{(u, v), v \in [\frac{1-x}{2}, \frac{1}{2}], u \in [v, 1 - v]\}.$$

Hence, for $x \in [0, \frac{1}{2}]$, tedious but elementary calculations lead to

$$\begin{aligned} \varphi(x) &:= \mathbb{E}[f_U(x)] = 4 \cdot \iint_{B_x} \frac{1}{v} - \frac{|x - u|}{v^2} \, du \, dv \\ &= -8(1 - x) \cdot \ln(1 - x) - 8x(1 - \ln(2)) \end{aligned}$$

By symmetry around $x = \frac{1}{2}$, the claim follows. \square

PROOF OF THEOREM 3.1: As usual, let $\eta_t(v)$ denote the opinion of individual $v \in \mathbb{Z}$ at time $t > 0$ and further let $f_t^{(v)}$ be the density corresponding to this random measure. For any fixed $\delta > 0$, let us define the random variables

$$F_t^{(v)}(\delta) := \eta_t(v)([0, \delta]) = \int_0^\delta f_t^{(v)}(x) \, dx, \quad \text{for all } t > 0, v \in \mathbb{Z}.$$

Their values lie in the interval $[0, 1]$, which actually coincides with the support of their distributions. Furthermore, we know that $\{F_0^{(v)}(\delta), v \in \mathbb{Z}\}$ are i.i.d. random variables and Fubini's theorem gives

$$\mathbb{E}[F_0^{(v)}(\delta)] = \int_0^\delta \varphi(x) \, dx. \quad (8)$$

We can disregard the case $\theta = 0$, since there won't be any dynamics and hence a.s. disagreement. Given $\theta \in (0, 1)$, define $\varepsilon := \frac{1}{2}(1 - \theta) > 0$ and choose $\delta > 0$ such that $\int_0^\delta \varphi(x) \, dx < \varepsilon$.

As mentioned above, the support of the distribution of $F_0^{(v)}(\delta)$ is $[0, 1]$ (without gaps), hence we can conclude as in Lemma 4.2 in [5] that any vertex is (one-sidedly) ε -flat with positive probability, with respect to the sequence $\{F_0^{(v)}(\delta), v \in \mathbb{Z}\}$. Due to $\mathbb{P}(F_0^{(v)}(\delta) = 1) = \delta^2 > 0$ and independence, the coincidence of the following two events occurs with positive probability for any $v \in \mathbb{Z}$:

- (a) Vertex $v - 1$ is ε -flat to the left and vertex $v + 1$ ε -flat to the right w.r.t. $\{F_0^{(v)}(\delta), v \in \mathbb{Z}\}$.
- (b) $F_0^{(v)}(\delta) = 1$

Using part (i) of Lemma 4.3 and the same line of reasoning as in the proof of Prop. 5.1 in [5], we can conclude that the edge $\langle v - 1, v \rangle$ – and similarly $\langle v, v + 1 \rangle$ – will be blocked forever, as

$$\begin{aligned} \|\eta_t(v) - \eta_t(v - 1)\|_{\text{TV}} &\geq |F_t^{(v)}(\delta) - F_t^{(v-1)}(\delta)| \\ &= F_0^{(v)}(\delta) - F_t^{(v-1)}(\delta) \\ &\geq 1 - (\mathbb{E} F_0^{(v-1)}(\delta) + \varepsilon) \\ &> 1 - 2\varepsilon = \theta. \end{aligned}$$

From the fact that approaching disagreement is shift-invariant, hence a 0-1-event, we can conclude that for $\theta \in [0, 1)$ there will a.s. be disagreement. \square

Remark

The trivial case $\theta = 1$ in Theorem 3.1 will surely not lead to blocked edges, so disagreement can be ruled out. However, this does not necessarily imply a consensus formation. The standard energy argument (as we will also use it in Lemma 6.4) fails, since the random symmetric triangular distribution does not have a finite second moment, i.e. $\mathbb{E}(f_U(x))^2 = \infty$ for all $x \in (0, 1)$.

Using the results for univariate opinions once more, we can however conclude consensus for $\theta = 1$ if we change to a different distance measure: the so-called Lévy-distance. Consider two probability distributions μ and ν on $[0, 1]$. Their *Lévy-distance* $\rho(\mu, \nu)$ is defined as the infimum of the set

$$\{\varepsilon > 0 \text{ s.t. } \mu([0, x - \varepsilon]) - \varepsilon \leq \nu([0, x]) \leq \mu([0, x + \varepsilon]) + \varepsilon \text{ for all } x \in [0, 1]\}.$$

To settle the case with $\theta = 1$ and ρ as distance measure, let us consider the univariate case, where $\{F_t^{(v)}(\delta) = \eta_t(v)([0, \delta]), t > 0, v \in \mathbb{Z}\}$ are the opinions assigned to the agents: As there is no bounded confidence restriction, any encounter leads to an update and the update rule (1) applies to both $\{\eta_t(v)\}_{v \in \mathbb{Z}}$ and $\{F_t^{(v)}(\delta)\}_{v \in \mathbb{Z}}$.

Hence, for any fixed $\delta \in [0, 1]$, from Theorem 4.1 and (8) we know that $F_t^{(v)}(\delta)$ converges to $\Phi(\delta) := \int_0^\delta \varphi(x) dx$ almost surely. Consequently, with probability 1, it holds

$$\lim_{t \rightarrow \infty} F_t^{(v)}(\delta) = \Phi(\delta) \quad \text{for all } v \in \mathbb{Z}, \delta \in [0, 1] \cap \mathbb{Q}.$$

Since all $F_t^{(v)}$ and Φ are continuous and increasing, this implies almost sure pointwise (in fact even uniform) convergence. In other words, for any $v \in \mathbb{Z}$ the opinion measure $\eta_t(v)$ converges with probability 1 *vaguely* to the intensity measure $\mathbb{E} \eta$, having density φ . As vague convergence of measures on a compact interval is metrized by the corresponding Lévy-metric (cf. for example Lemma 2 in [4]), this implies $\lim_{t \rightarrow \infty} \rho(\eta_t(v), \mathbb{E} \eta) = 0$ almost surely for all $v \in \mathbb{Z}$.

Note that a.s. consensus for $\theta = 1$ and the Lévy-metric does not immediately imply a result for the total variation case, as $\rho(\mu, \nu) \leq \|\mu - \nu\|_{\text{TV}}$ for two probability measures μ and ν .

6 Agents with bounded determination

In order to get a non-trivial phase transition in the parameter θ , let us now consider a situation in which all the agents feature at least a certain minimum of open-mindedness. This will be incorporated in our model by disallowing the initial random measure to be concentrated on a subinterval of length less than γ , for a fixed constant $\gamma \in (0, 1)$. We will refer to these as random *restricted triangular distributions*.

Before we can show the main result, Theorem 6.7, which states that there is a phase transition and the precise threshold value for the parameter θ , we need to study the altered intensity measure and verify a few auxiliary results, needed to guarantee the existence of ε -flat vertices (cf. Lemma 6.6).

Lemma 6.1

For fixed $\gamma \in (0, 1)$, consider the absolutely continuous random measure η_γ to be given by the density

$$f_U(x) = \frac{2}{|y-z|} \cdot \left(1 - \frac{2}{|y-z|} \cdot \left|x - \frac{y+z}{2}\right|\right)^+, \quad x \in [0, 1], \quad (9)$$

where $U = (y, z)$ is taken uniformly from the set $\{y, z \in [0, 1], |y - z| \geq \gamma\}$ and note that this corresponds to the expression in (3), conditional on the support being an interval of length at least γ . Then the density of its intensity measure $\mathbb{E} \eta_\gamma$ is given by the following expressions (assuming $0 \leq x \leq \frac{1}{2}$):

1) for $x \geq \gamma$

$$\varphi_\gamma(x) = -\frac{8}{(1-\gamma)^2} \left[(1-x) \ln(1-x) + x(1-\ln(2)) + \frac{\gamma}{4} \right],$$

2) for $x \leq 1 - \gamma$

$$\varphi_\gamma(x) = \begin{cases} -\frac{8}{(1-\gamma)^2} \left[(1-x) \ln \gamma + x + \frac{1-2x}{2\gamma} - \frac{\gamma}{2} \right], & x \leq \frac{\gamma}{2} \\ -\frac{8}{(1-\gamma)^2} \left[-x \ln(2x) + \ln \gamma + x + \frac{(1-x)^2 + x^2}{2\gamma} - \frac{3}{4}\gamma \right], & x \geq \frac{\gamma}{2} \end{cases},$$

3) for $x \leq \gamma$, $x \leq 1 - \gamma$

$$\varphi_\gamma(x) = \begin{cases} -\frac{8}{(1-\gamma)^2} \left[(1-x) \ln(1-x) + x - \frac{x^2}{2\gamma} \right], & x \leq \frac{\gamma}{2} \\ -\frac{8}{(1-\gamma)^2} \left[(1-x) \ln(1-x) - x \ln\left(\frac{2x}{\gamma}\right) + x + \frac{x^2}{2\gamma} - \frac{\gamma}{4} \right], & x \geq \frac{\gamma}{2} \end{cases}.$$

The corresponding expressions for $x \in [\frac{1}{2}, 1]$ are obtained by replacing x by $1-x$.

PROOF: As in the proof of Lemma 5.1, we can take U to be uniform on the set $A_\gamma := \{(y, z) \in \mathbb{R}^2, \gamma \leq y \leq 1, 0 \leq z \leq y - \gamma\}$ and consider the very same linear transform T , see Figure 4 below.

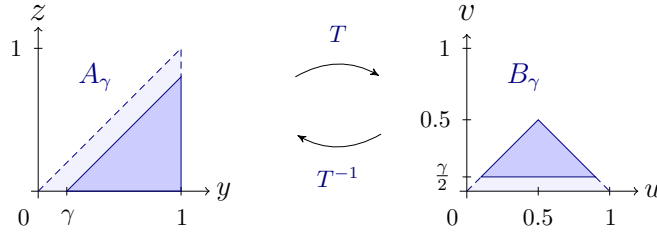


Figure 4: The restricted set A_γ forces a minimum amount of open-mindedness.

Then, $T(U)$ is uniform on the set $B_\gamma := \{(u, v), v \in [\frac{\gamma}{2}, \frac{1}{2}], u \in [v, 1-v]\}$ and the corresponding random density is still

$$f_{T(U)}(x) = \frac{1}{v} \cdot \left(1 - \frac{1}{v} \cdot |x - u|\right)^+, \quad x \in [0, 1].$$

Depending on the values of $x \in [0, \frac{1}{2}]$ and $\gamma \in (0, 1)$ – see Figure 5 for an illustration – quite cumbersome but nevertheless elementary calculations in the same vein as in the proof of Lemma 5.1 (which we will leave to the reader to perform) lead to the formulas stated above.

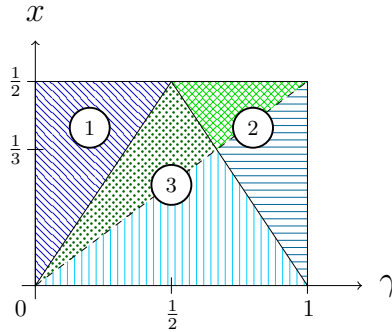


Figure 5: Different regimes for the form of $\varphi_\gamma(x)$.

The last claim follows again by the symmetry in x . □

Lemma 6.2

Consider $\mathbb{E}\eta_\gamma$ as in the previous lemma. Irrespectively of the value of $\gamma \in (0, 1)$, the density function $\varphi_\gamma(x)$, $x \in [0, 1]$, corresponding to the intensity measure, is (strictly) increasing on $[0, \frac{1}{2}]$ and (strictly) decreasing on $[\frac{1}{2}, 1]$.

PROOF: In principle, one could simply check the expressions for φ_γ given in Lemma 6.1. However, this simple fact can also be seen directly from the construction: Let us consider the random density in (9) to be generated by a vector $T = (U, V)$ that is chosen uniformly from B_γ , as described in the proof of Lemma 6.1. After having picked $V \sim \text{unif}([\frac{\gamma}{2}, \frac{1}{2}])$, we take U to be uniform on $[V, 1 - V]$.

Consider $x_1, x_2 \in [0, \frac{1}{2}]$, such that $x_1 < x_2$, and $V = v$ to be already fixed. First note that $f_T(x_1) < f_T(x_2)$ for $U > \frac{x_1+x_2}{2} \in (0, \frac{1}{2})$. If $v \leq \frac{x_1+x_2}{2}$, symmetry around $\frac{x_1+x_2}{2}$ shows that $f_{(U,v)}(x_1)$ and $f_{(U,v)}(x_2)$ have the same distribution for U conditioned on $[v, x_1 + x_2 - v]$. In conclusion, we found $f_T(x_1) < f_T(x_2)$ and especially

$$\varphi_\gamma(x_1) = \mathbb{E} f_T(x_1) < \mathbb{E} f_T(x_2) = \varphi_\gamma(x_2).$$

Symmetry of φ_γ around $x = \frac{1}{2}$ implies the second part of the claim. \square

Note that φ_γ can not be arbitrarily well approximated by the density of a restricted triangular distribution. Consequently, for $\varepsilon > 0$ sufficiently small, there can't be any ε -flat vertices with respect to the initial configuration as all triangular distributions have a positive distance to the intensity measure $\mathbb{E}\eta_\gamma$ bounded away from 0. For this reason, we have to go the same detour as in the proof of part (ii) of Thm. 2.2 in [6].

We need to verify that the density of the intensity measure actually can appear at a later time, more precisely be arbitrarily well approximated by the opinions that form when agents have interacted. This happens in fact for all positive values of the model parameter θ :

Lemma 6.3

Consider the Deffuant model on \mathbb{Z} with arbitrary parameter $\theta \in (0, 1]$ in which the initial opinions are i.i.d. absolutely continuous measures given by the random densities described in (9). Then, at any time $t > 0$ and for any $\varepsilon > 0$, a (sufficiently long) fixed finite section of the infinite path will hold opinions that are less than ε away from $\mathbb{E}\eta_\gamma$ in total variation distance (and be bounded by edges on which no Poisson events occurred up to time t) with positive probability.

PROOF: Fix $\theta \in (0, 1]$, $\varepsilon > 0$ and $t > 0$. The idea is to show that a set of agents with suitably assigned initial opinions can interact in such a way that at time t opinions close to $\mathbb{E}\eta_\gamma$ are formed.

Let us consider an i.i.d. sequence $(U_n)_{n \in \mathbb{N}}$ of random variables uniformly distributed on $A_\gamma = \{(y, z) \in \mathbb{R}^2, \gamma \leq y \leq 1, 0 \leq z \leq y - \gamma\}$, see Figure 4. Then we get the density corresponding to the initial opinion for agent $v \in \mathbb{N}$ by

$$f_0^{(v)} = f_{U_v} \tag{10}$$

where f_{U_v} is taken to be as in (9) and $f_t^{(v)}$ denotes the random density corresponding to the opinion of agent v at time t .

From Theorem 2.1 we know that with probability 1

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{v=1}^n f_0^{(v)}(x) = \varphi_\gamma(x) \quad (11)$$

uniformly in $x \in [0, 1]$.

It is not hard to check that $\max\{|y_1 - y_2|, |z_1 - z_2|\} \leq \delta$ entails

$$\frac{1}{2} \int_0^1 |f_{(y_1, z_1)}(x) - f_{(y_2, z_2)}(x)| \, dx \leq \frac{2\delta}{\gamma}, \quad (12)$$

in other words: If the coordinates of two vectors, (y_1, z_1) and (y_2, z_2) , shaping restricted triangular distributions in the sense of (9) do not differ by more than δ , the total variation distance between the corresponding measures is at most $\frac{2\delta}{\gamma}$.

Fix $m \geq \frac{16}{\gamma\theta}$ and subdivide $[0, 1]^2$ into m^2 squares. The standard SLLN implies that the fraction of $(U_n)_{n \in \mathbb{N}}$ landing in a square completely contained in A_γ a.s. tends to $\frac{2}{m^2(1-\gamma)^2} > 0$ as $n \rightarrow \infty$. We can therefore choose $N \in \mathbb{N}$ large enough such that, with positive probability, every square that is a subset of A_γ contains at least one of $(U_n)_{n=1}^N$ and

$$\left\| \frac{1}{N} \sum_{v=1}^N f_0^{(v)} - \varphi_\gamma \right\|_\infty \leq \varepsilon \quad (13)$$

Note that by symmetry under permutations, there is at least a chance of $\frac{1}{N!}$ that the agents $\{1, \dots, N\}$ are assigned these values from A_γ in such a way that those of neighboring agents do not differ much in both coordinates; more precisely, matching the values in a serpentine fashion as depicted in Figure 6 will keep discrepancies in the y -coordinate below $\frac{4}{m}$ and in the z -coordinate below $\frac{3}{m}$.

Putting things together, we found that for large enough N with non-zero probability the agents 1 through N have an initial configuration with a mean at total variation distance at most $\frac{\varepsilon}{2}$ to $\mathbb{E} \eta_\gamma$ and distance at most $\frac{8}{\gamma m} \leq \frac{\theta}{2}$ between neighbors.

Assume that there are no updates on the edges $\langle 0, 1 \rangle$ and $\langle N, N+1 \rangle$ up to time t . It is easy to check (by induction) that in this case, updates on the considered section in sweeps from left to right, i.e. first on $\langle 1, 2 \rangle$, then $\langle 2, 3 \rangle$ etc. until $\langle N-1, N \rangle$ repetitively, will keep the total variation distance of neighbors inside the section always below θ .

The following lemma finally verifies that a sufficiently large number of such sweeps will eventually bring the considered opinions within total variation distance $\frac{\varepsilon}{2}$ of their mean, due to the fact that the mean is preserved given that there are no updates on neither $\langle 0, 1 \rangle$ nor $\langle N, N+1 \rangle$.

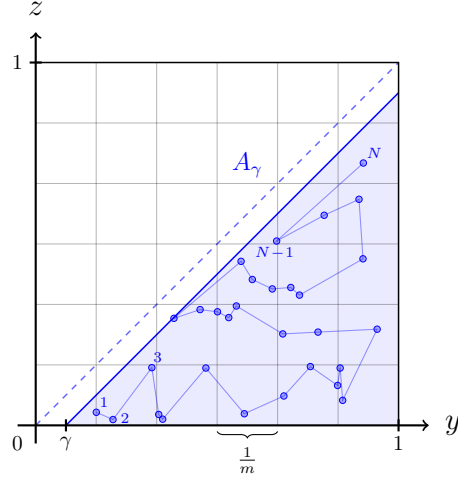


Figure 6: Finding suitable values for the initial opinions that can generate opinions close to the intensity measure $\mathbb{E} \eta_\gamma$ on a finite section.

Since the Poisson clocks and the initial configuration are independent, these two events coincide with positive probability and the claim is verified. \square

Lemma 6.4

If there are infinitely many (performed) updates along an edge, the total variation distance of the corresponding neighbors' opinions a.s. converges to 0.

PROOF: This statement follows immediately using the energy idea used in the proofs of Thm. 2.3 and 5.3 in [5]: Consider

$$W_t(v) := \int_0^1 [f_t^{(v)}(x)]^2 dx$$

to be the energy of vertex v at time t . When an update along the edge $\langle u, v \rangle$ is actually performed, i.e. the opinion values $(\eta_{t-}(u), \eta_{t-}(v))$ get replaced by $(\eta_t(u), \eta_t(v)) = ((1-\mu) \cdot \eta_{t-}(u) + \mu \cdot \eta_{t-}(v), (1-\mu) \cdot \eta_{t-}(v) + \mu \cdot \eta_{t-}(u))$ energy is lost to the amount of

$$\begin{aligned} 2\mu(1-\mu) \int_0^1 [f_{t-}^{(u)}(x) - f_{t-}^{(v)}(x)]^2 dx &\geq 2\mu(1-\mu) \left(\int_0^1 |f_{t-}^{(u)}(x) - f_{t-}^{(v)}(x)| dx \right)^2 \\ &= 8\mu(1-\mu) \|\eta_{t-}(u) - \eta_{t-}(v)\|_{TV}^2. \end{aligned}$$

As in Lemma 6.2 in [5], define the total energy $W_t^{\text{tot}}(v)$ at v to be $W_t(v)$ plus the energy lost on $\langle v, v+1 \rangle$ until time t . If we let $X(v)$ denote the random variable consisting of $\eta_0(v)$ and the Poisson process associated with the edge $\langle v, v+1 \rangle$, $\{X(v), v \in \mathbb{Z}\}$ is an i.i.d. sequence. $W_t^{\text{tot}}(0)$ is a measurable function of

$\{X(v), v \in \mathbb{Z}\}$ and $\{W_t^{\text{tot}}(v), v \in \mathbb{Z}\}$ its corresponding shifted equivalents. The well-known Pointwise Ergodic Theorem due to Birkhoff-Khinchin thus implies

$$\mathbb{E}[W_t^{\text{tot}}(0)] = \lim_{\substack{y \rightarrow -\infty \\ z \rightarrow \infty}} \frac{1}{z - y + 1} \sum_{u=y}^z W_t^{\text{tot}}(u) \quad \text{a.s.}$$

Note that there are a.s. infinitely many edges on which no Poisson event has occurred up to time t and that on a section between two such edges, the sum of total energies is preserved until t . Putting things together, we find

$$\begin{aligned} \mathbb{E}[W_t^{\text{tot}}(v)] &= \lim_{\substack{y \rightarrow -\infty \\ z \rightarrow \infty}} \frac{1}{z - y + 1} \sum_{u=y}^z W_t^{\text{tot}}(u) \quad \text{a.s.} \\ &= \lim_{\substack{y \rightarrow -\infty \\ z \rightarrow \infty}} \frac{1}{z - y + 1} \sum_{u=y}^z W_0^{\text{tot}}(u) \quad \text{a.s.} \\ &= \mathbb{E}[W_0^{\text{tot}}(v)] = \mathbb{E}[W_0(v)] = -\frac{8}{3(1-\gamma)} \left(1 + \frac{\ln(\gamma)}{1-\gamma}\right). \end{aligned}$$

If we assume for contradiction that with positive probability for some $\delta > 0$ the total variation distance $\|\eta_t(v) - \eta_t(v+1)\|_{\text{TV}}$ lies in $[\delta, \theta]$ for arbitrarily large t , the conditional Borel-Cantelli lemma (see e.g. Cor. 6.20 in [9]) forces infinitely many performed updates with the total variation distance being at least δ . As the total energy is always non-negative, this implies $\lim_{t \rightarrow \infty} \mathbb{E}[W_t^{\text{tot}}(v)] = \infty$, a contradiction. \square

Before we can use Lemma 6.3 to guarantee the existence of flat vertices at time $t > 0$, we need to check that (11) also holds for time $t > 0$.

Lemma 6.5

Given the Deffuant model as described in Lemma 6.3, for all $t \geq 0$, it holds that

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{v=1}^n f_t^{(v)} = \varphi_\gamma$$

almost surely with respect to the supremum norm on $[0, 1]$.

PROOF: Fix $t > 0$. From Theorem 2.1 we know that

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{v=1}^n f_0^{(v)} = \varphi_\gamma \quad \text{a.s.}$$

with respect to the supremum norm. Using the fact that the densities are uniformly bounded by $\frac{2}{\gamma}$ we can conclude that this convergence holds even for $t > 0$, by the same token as in [6]:

To the right of site 1, a.s. there is an infinite increasing sequence of nodes $(v_k)_{k \in \mathbb{N}}$, such that there was no Poisson event up to time t on the collection of

edges $\{\langle v_k, v_k + 1 \rangle, k \in \mathbb{N}\}$. We denote the random lengths of the intervals in between by $L_k := v_{k+1} - v_k$, for $k \in \mathbb{N}$. In addition, let $L_0 := v_1 - v_0$ be the length of the interval including agent 1, where $\langle v_0, v_0 + 1 \rangle$ is the first edge to the left of site 1 without Poisson event. Independence of the involved Poisson processes entails that $(L_k)_{k \in \mathbb{N}_0}$ is an i.i.d. sequence of random variables having geometric distribution on \mathbb{N} with parameter e^{-t} .

Fix $\delta > 0$. Using the Borel-Cantelli lemma we find that the event

$$E_\delta := \{L_0 = \infty\} \cup \limsup_{k \rightarrow \infty} \left\{ L_k \geq k \cdot \frac{\delta\gamma}{2} \right\}$$

has probability 0.

The Deffuant model is mass-preserving in the sense that the sum of opinions of two interacting agents is always preserved. Therefore it holds for all $k \in \mathbb{N}$:

$$\sum_{u=v_0+1}^{v_k} f_0^{(v)} = \sum_{u=v_0+1}^{v_k} f_t^{(u)}.$$

Furthermore, for some $v \in \{v_k + 1, \dots, v_{k+1}\}$, the event

$$\left\| \frac{1}{v - v_0} \sum_{u=v_0+1}^v f_0^{(u)} - \frac{1}{v - v_0} \sum_{u=v_0+1}^v f_t^{(u)} \right\|_\infty \geq \delta$$

forces $L_k \geq k \cdot \frac{\delta\gamma}{2}$, since $v_k \geq k$ and the density $f_s^{(u)}$ is non-negative and uniformly bounded by $\frac{2}{\gamma}$ for all $u \in \mathbb{Z}$ and times $s \geq 0$.

In conclusion, given E_δ^c , it holds that

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{v=1}^n f_t^{(v)}(x) &= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{v=v_0+1}^n f_t^{(v)}(x) \leq \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{v=v_0+1}^n f_0^{(v)}(x) + \delta \\ &= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{v=1}^n f_0^{(v)}(x) + \delta = \varphi_\gamma(x) + \delta \end{aligned}$$

uniformly in $x \in [0, 1]$. In the same way we get $\varphi_\gamma(x) - \delta$ as a lower bound and letting δ go to 0 finally verifies

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{v=1}^n f_t^{(v)} = \varphi_\gamma \quad \text{a.s.} \quad (14)$$

w.r.t. the supremum norm. \square

Lemma 6.6

Given the Deffuant model as described in Lemma 6.3 and $\varepsilon > 0$, the following holds for all $t > 0$:

- (i) With non-zero probability, there has been no Poisson event on the edge $\langle 0, 1 \rangle$ until time t and site 1 is ε -flat to the right with respect to the configuration $\{\eta_t(v)\}_{v \in \mathbb{Z}}$ and distance measure $\|\cdot\|_{\text{TV}}$.

(ii) With non-zero probability, site 0 is two-sidedly ε -flat with respect to the configuration $\{\eta_t(v)\}_{v \in \mathbb{Z}}$ and distance measure $\|\cdot\|_{\text{TV}}$.

PROOF: In order to verify these claims, we only have to put together the ingredients established in Lemmas 6.3 and 6.5. As in the proof of Thm. 2.2 in [6], we will do this by using a conditional variant of the coupling technique introduced in [12], that became known as *local modification* in percolation theory.

Fix $\varepsilon > 0$. Recall that for absolutely continuous measures μ and ν on $[0, 1]$, with densities f and g respectively, we get $\|\mu - \nu\|_{\text{TV}} \leq \frac{1}{2} \cdot \|f - g\|_\infty$ and let B denote the following event:

$$\left\{ \left\| \frac{1}{n} \sum_{v=1}^n \eta_t(v) - \mathbb{E} \eta_\gamma \right\|_{\text{TV}} \leq \frac{\varepsilon}{3} \text{ and } \left\| \frac{1}{n} \sum_{v=-n}^{-1} \eta_t(v) - \mathbb{E} \eta_\gamma \right\|_{\text{TV}} \leq \frac{\varepsilon}{3}, \text{ for all } n \geq N \right\}.$$

From Lemmas 6.3 and 6.5, we know that $N \in \mathbb{N}$ can be chosen sufficiently large such that $\mathbb{P}(B) > 1 - e^{-2t}$ and $\mathbb{P}(C \cap D) > 0$, where C denotes the event of no Poisson events on the two edges $\langle 0, 1 \rangle$ and $\langle N, N+1 \rangle$ up to time t ,

$$D := \left\{ \|\eta_t(v) - \mathbb{E} \eta_\gamma\|_{\text{TV}} \leq \frac{\varepsilon}{3} \text{ for all } 1 \leq v \leq N \right\}.$$

Additionally, since $\mathbb{P}(C) = e^{-2t}$, we must have $\mathbb{P}(B \cap C) > 0$.

Now let $\boldsymbol{\eta}_t := \{\eta_t(v)\}_{v \in \mathbb{Z}}$ and $\boldsymbol{\eta}'_t := \{\eta'_t(v)\}_{v \in \mathbb{Z}}$ be the configurations at time t originated from two independent copies of the considered model. There is a strictly positive probability that $B \cap C$ happens for $\boldsymbol{\eta}_t$ as well as that $C \cap D$ happens for $\boldsymbol{\eta}'_t$. Given C , the hybrid process $\tilde{\boldsymbol{\eta}}_t$ defined by

$$\tilde{\eta}_t(v) = \begin{cases} \eta_t(v) & \text{if } v \notin \{1, \dots, N\} \\ \eta'_t(v) & \text{if } v \in \{1, \dots, N\} \end{cases}$$

is a perfectly fine copy of the model as well, showing that the event $B \cap D$ has non-zero probability. It is an easy exercise to check that $B \cap D$ actually implies the ε -flatness to the right of site 1.

In fact, the same argument applies to the second setting. Here, however, we choose C to be the event that there were no Poisson events on $\langle -N-1, -N \rangle$ and $\langle N, N+1 \rangle$ as well as $D := \left\{ \|\eta_t(v) - \mathbb{E} \eta_\gamma\|_{\text{TV}} \leq \frac{\varepsilon}{3} \text{ for all } -N \leq v \leq N \right\}$. Then the two-sidedly ε -flatness of site 0 follows from $B \cap D$. \square

Let us now use Lemmas 6.4 and 6.6 to prove the main statement about the model featuring restricted random triangular distributions.

Theorem 6.7

Consider the Deffuant model on \mathbb{Z} , in which the total variation distance is used to measure the difference between two opinions and in which the initial opinions are given by independently assigned random restricted triangular distributions with fixed $\gamma \in (0, 1)$ as described in (9). Then there is a sharp phase transition in the following sense: for $\theta \in [0, \theta_c)$, the system almost surely approaches disagreement in the long run; for $\theta \in (\theta_c, 1]$, it almost surely approaches consensus.

The threshold θ_c is given by

$$\begin{aligned}\theta_c &= \frac{1}{2} \int_0^1 \left| f_{(\gamma,0)}(x) - \varphi_\gamma(x) \right| dx \\ &= \frac{1}{2} \int_0^1 \left| \frac{2}{\gamma} \cdot \left(1 - \frac{2}{\gamma} \cdot \left|x - \frac{\gamma}{2}\right|\right)^+ - \varphi_\gamma(x) \right| dx.\end{aligned}\tag{15}$$

PROOF: As mentioned in Section 4, we will closely follow the ideas in [5] to establish this result, just now the opinions are given by (random) absolutely continuous measures, or rather their density functions. In fact, most of the work has already been done by showing Lemma 6.6. Let us define

$$\theta_c := \frac{1}{2} \int_0^1 \left| f_{(\gamma,0)}(x) - \varphi_\gamma(x) \right| dx$$

and $\varepsilon := \frac{1}{\gamma} \cdot |\theta_c - \theta|$. In the sequel, we will consider the two regimes: the subcritical one ($\theta < \theta_c$) and the supercritical one ($\theta > \theta_c$).

In the subcritical regime, fix $t > 0$ and let B denote the event that there are no Poisson events neither on $\langle -1, 0 \rangle$ nor on $\langle 0, 1 \rangle$ during $[0, t]$ and site -1 is ε -flat to the left, site 1 is ε -flat to the right with respect to the configuration $\{\eta_t(v)\}_{v \in \mathbb{Z}}$. By Lemma 6.6 part (i), the obvious symmetry and conditional independence, we know that B occurs with positive probability. Let the initial opinion of agent 0 be given by f_{U_0} in the sense of (10) and C be the event that the first coordinate of U_0 is less than $\gamma + \frac{\varepsilon\gamma}{2}$, which by the shape of A_γ (see Figure 4) and (12) entails

$$\|\eta_0(0) - \mathbb{E} \eta_\gamma\|_{\text{TV}} \geq \theta_c - \varepsilon.$$

Given that there are no Poisson events on edges incident to site 0 , we can (again by local modification) conclude that $B \cap C$ has positive probability. From Lemma 4.3, we know that the total variation distance between $\eta_s(1)$ and the intensity measure $\mathbb{E} \eta_\gamma$ will not exceed ε for $s \geq t$ due to its one-sided ε -flatness if there is no interaction with site 0 (same for $\eta_s(-1)$). However, given $B \cap C$ the opinions $\eta_t(1)$ and $\eta_t(0) = \eta_0(0)$ are at distance larger than $\theta_c - 2\varepsilon > \theta$ and hence they never will be close enough to interact, since the same holds for $\eta_t(-1)$, which leaves the opinion at site 0 unchanged for all time. In other words, with non-zero probability the edge $\langle 0, 1 \rangle$ will be finally blocked (same for $\langle -1, 0 \rangle$).

To conclude the claimed almost sure behavior, we apply the ergodicity argument used in the proof of Lemma 6.4 once again: Whether the configuration approaches disagreement or not can be checked given the initial configuration plus all Poisson processes associated to the edges. The sequence $\{X(v), v \in \mathbb{Z}\}$ (as defined in the proof of Lemma 6.4) is i.i.d., hence ergodic with respect to shifts. Thus the translation-invariant event “disagreement” necessarily has to be trivial, i.e. must have probability either 0 or 1 . Since we already showed that its probability is non-zero, the event has to be an almost sure one in the subcritical regime.

In the supercritical case, we know that at time $t > 0$ any fixed site is two-sidedly ε -flat with positive probability (part (ii) of Lemma 6.6). Lemma 6.2 implies that the largest total variation distance of a restricted triangular distribution as defined in (9) to the intensity measure $\mathbb{E}\eta_\gamma$ is given by what we defined as θ_c . Since all opinions at a later time are convex combinations of the initial ones, this distance can not be exceeded. If site 0 is two-sidedly ε -flat with respect to the configuration $\{\eta_t(v)\}_{v \in \mathbb{Z}}$, the opinion $\eta_s(0)$ will be at total variation distance at most 6ε to $\mathbb{E}\eta_\gamma$, for all $s \geq t$ (part (ii) of Lemma 4.3). This implies that its neighboring opinions differ by not more than $\theta_c + 6\varepsilon < \theta$, hence Lemma 6.4 forces the differences to converge to 0. By induction, this is actually true for any pair of neighbors. Again, ergodicity ensures that this consensus behavior occurs with probability 1.

It further implies the almost sure existence of two-sidedly ε -flat vertices for any strictly positive value of ε . For this reason, the measure, the opinions converge to, must be the intensity measure, which concludes the proof. \square

Example 6.8

Let us consider the Deffuant model on \mathbb{Z} with opinions being absolutely continuous probability distributions and the initial ones given by random restricted triangular distributions with parameter $\gamma = \frac{1}{3}$. From Lemma 6.1 we know that the corresponding intensity measure has the somewhat cumbersome density

$$\varphi_{\frac{1}{3}}(x) = \begin{cases} -18 \left[(1-x) \ln(1-x) - \frac{3}{2} x^2 + x \right], & 0 \leq x \leq \frac{1}{6} \\ -18 \left[(1-x) \ln(1-x) - x \ln(6x) + x + \frac{3}{2} x^2 - \frac{1}{12} \right], & \frac{1}{6} \leq x \leq \frac{1}{3} \\ -18 \left[(1-x) \ln(1-x) + x(1 - \ln(2)) + \frac{1}{12} \right], & \frac{1}{3} \leq x \leq \frac{1}{2} \\ -18 \left[x \ln(x) + (1-x)(1 - \ln(2)) + \frac{1}{12} \right], & \frac{1}{2} \leq x \leq \frac{2}{3} \\ -18 \left[x \ln(x) - (1-x) \ln(6-6x) + \frac{3}{2} x^2 - 4x + \frac{29}{12} \right], & \frac{2}{3} \leq x \leq \frac{5}{6} \\ -18 \left[x \ln(x) - \frac{3}{2} x^2 + 2x - \frac{1}{2} \right], & \frac{5}{6} \leq x \leq 1. \end{cases}$$

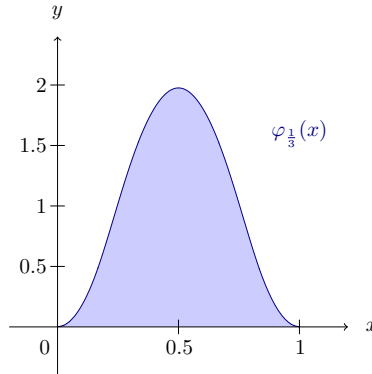


Figure 7: The intensity measure is concentrated more towards the center if the random triangular distributions are restricted to a minimum width.

If the total variation distance is used to measure the disparity of two opinions, we can conclude from Theorem 6.7 that the threshold θ_c for this model takes on the value

$$\theta_c = \frac{1}{2} \int_0^1 \left| f_{(0, \frac{1}{3})}(x) - \varphi_{\frac{1}{3}}(x) \right| dx \approx 0.83172.$$

7 Alternative choices (concerning initial configuration and distance measure) and inhomogeneous open-mindedness

No doubt that the extension of the Deffuant model on \mathbb{Z} to measure-valued opinions leaves a wide range of possible laws for the initial configuration to be examined. We saw trivial behavior for triangular distributions and a non-trivial phase transition for restricted triangular distributions.

If we want to stick to absolutely continuous measures on a compact support S and $\|\cdot\|_{\text{TV}}$ as distance measure, the line of argument from Section 6 will in principle carry over on condition that

$$\mathbb{E} \int_S [f_\omega(x)]^2 dx < \infty,$$

where f_ω , $\omega \in \Omega$, denotes the random initial density shaped by a probability space (Ω, \mathbb{P}) , and that the total variation distance of an initial opinion to the intensity measure is a.s. bounded away from 1. The first condition is needed to prove Lemma 6.4, the latter will in fact give the threshold θ_c , as essential supremum of the total variation distance of initial opinions to the intensity measure. There is however one issue, that must not be overlooked: In order to establish Lemma 6.3, we needed that there are no major gaps in the support of the initial opinions – just as in the case of finite-dimensional opinions. To examine this problem more closely, elaborate geometric considerations as in [7] seem to be necessary and we will thus leave this for future studies.

If one wants to include point processes as opinions, in many situations the total variation distance will not work as a meaningful measure for the discrepancy of two opinions, at least if the support of two such processes is disjoint with positive probability. Using the Lévy-distance instead could however lead to interesting models in such a setting. In fact, even in the case of triangular distributions, it seems to be unrealistic that two determined agents are at maximal distance, whether the intervals, on which their opinions are concentrated, are in close proximity or at different ends of the spectrum. From this point of view, albeit more difficult to handle, the Lévy-distance appears to be a more suitable choice.

Finally, it might be interesting to point out the new feature of the model (compared to finite-dimensional opinions) that was mentioned in the beginning of Section 5 once more and put it into a broader context: In the Deffuant model with triangular distributions, besides the common tolerance parameter θ and

willingness to compromise μ , we have a diversified scale of open-mindedness of the agents shaped by the random support of their initial opinion.

There have been attempts, e.g. by Deffuant et al. in [14], to simulate the long-term behavior of a variant of the model in which the agents have different θ -values. On the analytical side, this is quite a crucial change since it brings along situations in which the opinion of only one of two interacting neighbors is updated. Then the sum of opinions is no longer preserved, which renders void many of our central arguments. Another advance in the same direction is the so-called *relative agreement model*, introduced in [2]. There, the bounded confidence rule is dropped and replaced by a continuous counterpart: Agents feature both a real-valued opinion and a separate value corresponding to their individual uncertainty, which taken together shape a dispersed opinion in the form of a symmetric interval of length two times the uncertainty around the opinion value. If an agent gets influenced by another, the impact depends on the overlap of the two opinion intervals relative to the length of the interval corresponding to the influenced agent. Again, the asymmetric way of updating opinion values makes the relative agreement model, although based on the same principles and ideas, qualitatively quite different.

On the modelling side, the way inhomogeneous open-mindedness or uncertainty is incorporated in our extension of the model does not only avoid this issue, but also lead to the realistic property that agents themselves become more open-minded by interacting with open-minded neighbors.

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